Lecture 17

Wednesday, March 4, 2020 5:25 AM

Recall. $\Omega \subseteq \mathbb{C}^n$

Domain of holomorphy. TFAE:

(i) $K \subset\subset \Omega \Rightarrow \hat{K}_\Omega \subset\subset \Omega$.

(ii) $\exists f \in \mathcal{O}(\Omega)$ that do not extend across any $\zeta \in \partial\Omega$.

Pseudoconvexity. TFAE:

(i) $-\log \delta(z, \mathbb{C}^n \setminus \Omega)$ is PSH.

(ii) $K \subset\subset \Omega \Rightarrow \hat{K}^P_\Omega \subset\subset \Omega$.

(iii) $\exists u \in PSH \cap \mathcal{C}$ s.t. $\overline{\Omega}_c = \{z \in \Omega : u(z) < c\} \subset\subset \Omega$, (PSH exhaustion fcn).

Have shown: Ω d.o.hol. $\Rightarrow \Omega$ ψcvx. Converse is true (Levi problem), but we have not yet established this.

Now, continue to investigate ψcvxty.

We shall show that pseudoconvexity is local property near $\partial\Omega$.

Thm1. Let $\Omega \subseteq \mathbb{C}^n$. If $\forall \zeta \in \partial\Omega$ $\exists$ open nbhd $U_\zeta \subseteq \mathbb{C}^n$ s.t. $\Omega \cap U_\zeta$ is ψcvx, then Ω is ψcvx.

Pf. For each $\zeta \in \partial\Omega$ $\exists$ $U'_\zeta \subseteq U_\zeta$ s.t. $\delta(z, \mathbb{C}^n \setminus \Omega) = \delta(z, \mathbb{C}^n \setminus (\Omega \cap U_\zeta))$, for $z \in \Omega \cap U'_\zeta$.

By assumption, $-\log \delta(z, \mathbb{C}^n \setminus (\Omega \cap U_\zeta))$ is PSH in $\Omega \cap U_\zeta \Rightarrow$

$u(z) := -\log \delta(z, \mathbb{C}^n \setminus \Omega)$ is PSH in $\Omega \cap U'_\zeta$. $\Rightarrow \exists$ closed $F \subseteq \Omega$ s.t. u is PSH in $\Omega \setminus F$. Let φ be smooth, convex $\nearrow$ fcn of $|z|^2$ s.t. $\varphi(|z|) > u(z)$ on F. (Consider $M(r) = \sup_{z \in F, |z| \le r} u(z)$. Then $M(r) \nearrow$. by $\nearrow$ + convex (Ex.)

Now, let $\varphi(r)$ be smooth convex majorant.) Since $\varphi \in PSH(\mathbb{C}^n)$, $\varphi > u$ on

$\varphi(|z|) > u(z)$ on F. ([illegible] $z \in F, |z|=1$ by $\nearrow$ + convex (Ex.)

Now let $\varphi(r)$ be $\overset{\text{smooth}}{\text{convex majorant}}$.) Since $\varphi \in PSH(\mathbb{C}^n)$, $\varphi > u$ on open nbhd of F, the fcn $v = \max(u, \varphi)$ is $PSH \cap \mathcal{C}$ in Ω. Clear, $\Omega_c := \{z \in \Omega : v(z) < c\}$ is precompact in Ω (i.e. satisfies (iii) above) $\Rightarrow$ Ω is Ψcvx. ∎

Locality of Ψcvxty is even more clear if $\partial\Omega$ is smooth.

<u>Thm 2</u>. Let $\Omega \subseteq \mathbb{C}^n$ and $\partial\Omega \in \mathcal{C}^2$ and let $\rho \in \mathcal{C}^2$ in open nbhd of $\overline{\Omega}$, $\Omega = \{\rho < 0\}$ and $d\rho|_{\partial\Omega} \neq 0$. Then, Ω is Ψcvx $\iff$ $\forall \zeta \in \partial\Omega$ and $w \in \mathbb{C}^n$ s.t.

$$\underset{①}{\sum_i \rho_{z_i}(\zeta) w^i = 0} \Rightarrow \underset{②}{\sum_{i,j} \rho_{z_i \bar{z}_j}(\zeta) w^i \overline{w^j} \geq 0} \quad \text{(Levi condition)}.$$

<u>Rem.</u> • Using vector field, form notation: $X = \sum_{j=1}^n w^j \frac{\partial}{\partial z_j}$.

① : $\partial\rho|_\zeta(X) = 0$ or $X \in T^{1,0}_\zeta \partial\Omega \subseteq \mathbb{C} \otimes T_\zeta \partial\Omega$

② : $i\partial\bar{\partial}\rho|_\zeta(X, \bar{X}) \geq 0$; <u>Levi form</u> of $\partial\Omega$ positive semi definite.

• <u>Ex:</u> Easy to see that $T^{1,0}_\zeta \partial\Omega$ indep. of ρ, as is Levi form condition (2).

<u>Pf.</u> We start with Ψcvxity $\Rightarrow$ Levi condition.

By remark, we are free to choose defining fcn $\rho \in \mathcal{C}^2$ and verify Levi cond. Take:

$$\rho = \begin{cases} -\delta(z, \mathbb{C}^n \setminus \Omega), & z \in \Omega \\ 0, & z \in \partial\Omega \\ \delta(z, \Omega), & z \in \mathbb{C}^n \setminus \Omega. \end{cases}$$

One can show this ρ is $\mathcal{C}^2$ near $\partial\Omega$ when $\partial\Omega$ is $\mathcal{C}^2$ (i.e. $\exists$ some $\rho \in \mathcal{C}^2$). Now, $-\log \delta(z, \mathbb{C}^n \setminus \Omega) = -\log(-\rho)$ is PSH in Ω. Thus, with $u = -\log(-\rho)$ we compute

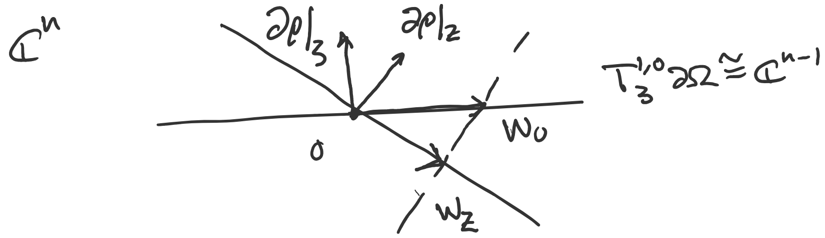
$$u_{z_i} = -\frac{1}{\rho}\rho_{z_i}, \quad u_{z_i \bar{z}_j} = \frac{-\rho_{z_i \bar{z}_j}\rho + \rho_{z_i}\rho_{\bar{z}_j}}{\rho^2} \underset{PSH}{\Rightarrow} \forall z \in \Omega, w \in \mathbb{C}^n$$

$u_{z_i} = -\frac{1}{\rho}\rho_{z_i}$, $u_{z_i\bar{z}_j} = \frac{(\ldots)}{\rho^2}$ $\Rightarrow$ $\forall z \in \Omega, w \ldots$ PSH

$$0 \le \sum_{i,j}\Big[\underbrace{(-\rho)}_{>0 \text{ in } \Omega}\rho_{z_i\bar{z}_j} w^i\overline{w^j} + \rho_{z_i}w^i\overline{\rho_{z_j}w^j}\Big] \Rightarrow$$

$$\sum_i \rho_{z_i} w^i = 0 \Rightarrow \sum_{i,j}\rho_{z_i\bar{z}_j} w^i\overline{w^j} \geq 0. \quad (3)$$

Pick $\zeta \in \partial\Omega$, $w_0 \in T^{1,0}_\zeta M$ $\overset{\#}{0}$ (i.e. $\sum_i \rho_{z_i}(\zeta) w_0^i = 0$). Let $z \in \Omega$, $z \to \zeta$. Since $\partial\rho|_\zeta \neq 0$, we can find $w_z \in \mathbb{C}^n$ s.t. $w_z \to w_0$ as $z \to \zeta$.



Letting $z \to \zeta$, $w_z \to w_0$ in (3) $\Rightarrow$ Levi condition.

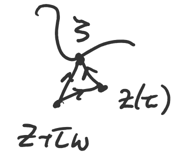
For the converse, suppose $-\log\delta(z, \mathbb{C}^n \setminus \Omega)$ is <u>not PSH</u> near $\partial\Omega$ (suffices to check near $\partial\Omega$ by Thm 1 above). For simplicity here, we choose δ to be a norm on $\mathbb{C}^n$ s.t. we have Δ-ineq. We can then find $z \in \Omega$ near $\partial\Omega$ (so that $\delta(z, \mathbb{C}^n \setminus \Omega)$ is C^2), $w \in \mathbb{C}^n$ s.t. $\frac{\partial^2}{\partial\tau\partial\bar{\tau}}\left(-\log\delta(z+\tau w, \mathbb{C}^n\setminus\Omega)\right)\Big|_{\tau=0} = -c < 0$. Choose $a \in \mathbb{C}^n$ s.t. $\delta(z, \mathbb{C}^n \setminus \Omega) = \delta(a)$ ($\Rightarrow z + a \in \partial\Omega$), and Taylor expand at $\tau = 0$:

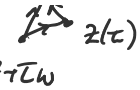
$$\log\delta(z+\tau w, \mathbb{C}^n\setminus\Omega) = \log\delta(a) + \operatorname{Re}(A\tau + B\tau^2) + c|\tau|^2 + o(|\tau|^2) \Rightarrow$$

$$\delta(z+\tau w, \mathbb{C}^n\setminus\Omega) \geq \delta(a)\, e^{c|\tau|^2/2}\left|e^{A\tau + B\tau^2}\right| \quad \text{for } |\tau| \leq \varepsilon.$$

Consider the holomorphic disk $z(\tau) = z + \tau w + a e^{A\tau + B\tau^2}$, $|\tau| \leq \varepsilon$.

We have by Δ-ineq.



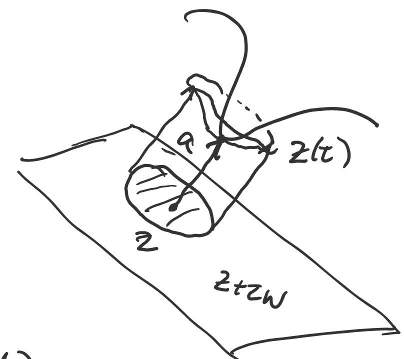


$$\delta(z(\tau), \mathbb{C}^n \setminus \Omega) \geq \delta(z+\tau w, \mathbb{C}^n \setminus \Omega) - \delta(a e^{A\tau + B\tau^2}) \geq$$
$$\delta(a) e^{c|\tau|^2/2} |e^{A\tau+B\tau^2}| - |e^{A\tau+B\tau^2}| \delta(a)$$
$$= \delta(a)\left(e^{c|\tau|^2/2} - 1\right)|e^{A\tau+B\tau^2}| \qquad (*)$$

Since $\delta(z(0), \mathbb{C}^n \setminus \Omega) = \delta(z+a) = 0 \overset{(*)}{\Rightarrow} \frac{\partial}{\partial \tau} \delta(z(\tau), \mathbb{C}^n \setminus \Omega)\Big|_{\tau=0} = 0$

and $\frac{\partial^2}{\partial \tau \partial \bar{\tau}} \delta(z(\tau), \mathbb{C}^n \setminus \Omega) > 0$.

Ex.



(violating conf. princ.)

With ρ as above

Compute: $0 = \frac{\partial}{\partial \tau} \rho(z(\tau))\Big|_{\tau=0} = \sum_i \frac{\partial \rho}{\partial z_i}(\underbrace{z(0)}_{\in \partial\Omega}) z_i'(0)$

And: $0 > \frac{\partial^2}{\partial \tau \partial \bar{\tau}} \rho(z(\tau)) = \{\tau \to z(\tau) \text{ holom.}\} = \sum_{i,j} \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \rho(z(0)) z_i'(0) \overline{z_j'(0)}$

Thus, we have violated the Levi condition, so Levi cond. $\Rightarrow \varphi$ cvx.

$\blacksquare$